Lower bound on PPT distillable entanglement from isotropic states

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1. **Relevant paper:**

   Eric M. Rains - *A semidefinite program for distillable entanglement*

   (https://arxiv.org/abs/quant-ph/0008047)

2. **Theorems chosen and their statements:**

   Theorem 6.1:

   Let $D_\Gamma(\rho)$ be the PPT distillable entanglement of the state ρ defined as

   $$D_\Gamma(\rho) := \sup\left\{r : \lim_{n\to\infty} F_\Gamma(\rho^{\otimes n}; \lfloor 2^{rn} \rfloor) = 1\right\},$$

   where $F_\Gamma(\rho; K)$ is the fidelity of K-state PPT distillation defined by

   $$F_\Gamma(\rho; K) = \max_{\Psi} F(\Psi(\rho)),$$

   where Ψ ranges over all PPT operations from $(\mathcal{V} \otimes \mathcal{W})$ to $(\mathbb{C}^K \otimes \mathbb{C}^K)$ for complex Euclidean spaces $\mathcal{V}$ and $\mathcal{W}$.

   Let $I_d(f)$ denote the isotropic state

   $$I_d(f) := f\Phi(d) + \frac{1-f}{d^2-1}(\mathbb{1} - \Phi(d))$$

   of dimension d and fidelity f, where $\Phi(d)$ is the maximally entangled state of dimension d defined as

   $$\Phi(d) = \frac{1}{d} \sum_{1\leq i,j \leq d} (e_i \otimes e_i)(e_j \otimes e_j)^*.$$

   Fix a fidelity $\frac{1}{2} \leq f \leq 1$ and an integer $d > 1$. Then

   $$D_\Gamma(I_d(f)) \geq \max(\log_2 d + f \log_2 f + (1-f)\log_2 \frac{1-f}{d+1}, 0).$$

   Theorem 3.1:

   For any state ρ and any positive integer K,

   $$F_\Gamma(\rho; K) = \max_F \mathrm{Tr}(F\rho),$$

   where F ranges over Hermitian operators such that

   $$0 \leq F \leq 1, \; -\frac{1}{K} \leq F^\Gamma \leq \frac{1}{K},$$

   where F^Γ denotes the partial transpose of F.

3. **Proof of Theorem (6.1)**

Proof strategy

The Theorem (6.1) gives the lower bound on distillable entanglement D_Γ of the isotropic state $I_d(f)$. To derive the lower bound and thus prove the Theorem, we will construct a Hermitian operator $F_n(w)$ which meets the requirements stated in Theorem (3.1). In that Theorem, the fidelity of distillation F_Γ is defined as the maximum of the $\mathrm{Tr}(F\rho)$ taken over all Hermitian operators F which fulfill the conditions. By finding the feasible operator, we can calculate the lower bound on the fidelity F_Γ. Then, by considering the limit as $(n \to \infty)$, we will calculate the maximal rate of distillation r for which the fidelity F_Γ tends to 1. The expression obtained for such a rate will be the lower bound on the distillable entanglement D_Γ from the isotropic state $I_d(f)$.

Proof

Let us define the set of orthogonal projection $\Pi_{\Phi(d)} = \{\Phi(d), \mathbb{1} - \Phi(d)\}$, where $\Phi(d)$ is the d-dimensional maximally entangled state

$$\Phi(d) = \frac{1}{d} \sum_{1 \leq a,b \leq d} (e_a \otimes e_a)(e_b \otimes e_b)^* = \frac{1}{d} \sum_{a,b} E_{a,b} \otimes E_{a,b},$$

where $\{e_i\}$ is the standard orthonormal basis in the complex Euclidean space $\mathcal{X} = \mathbb{C}^d$ where $d > 1$. We will consider the set $\mathcal{P}_n$ which consists of all tensor products of the form

$$P = P_i^{\otimes 1 \leq i \leq n},$$

where $P_i \in \Pi_{\Phi(d)}$. For example

$$\mathcal{P}_2 = \{\Phi(d) \otimes \Phi(d), \Phi(d) \otimes (\mathbb{1} - \Phi(d)), (\mathbb{1} - \Phi(d)) \otimes (\mathbb{1} - \Phi(d)), (\mathbb{1} - \Phi(d)) \otimes \Phi(d)\}.$$

Elements of $\mathcal{P}_n$ are mutually orthogonal projections. Let us denote the partial transpose operation with the superscript Γ. Then,

$$\Phi(d)^\Gamma = \left[\frac{1}{d} \sum_{a,b} E_{a,b} \otimes E_{a,b}\right]^\Gamma = \frac{1}{d} \sum_{a,b} E_{b,a} \otimes E_{a,b} = \frac{W}{d},$$

where W is the swap operator and

$$\mathbb{1} - \Phi(d)^\Gamma = \mathbb{1} - \frac{W}{d} = \frac{d\mathbb{1} - W}{d}.$$

Therefore

$$|\Phi(d)^\Gamma| = \frac{|W|}{d} = \frac{\mathbb{1}}{d},$$

since eigenvalues of W form a set $\{+1, -1\}$ and

$$|\mathbb{1} - \Phi(d)^\Gamma| = \frac{|d\mathbb{1} - W|}{d},$$

$$\frac{(d-1)\mathbb{1}}{d} \leq |\mathbb{1} - \Phi(d)^{\Gamma}| \leq \frac{(d+1)\mathbb{1}}{d}.$$

When we consider operators $P \in \mathcal{P}_n$, then the following bound can be established for their partial transpositions

$$|P^{\Gamma}| \leq d^{-n}(d+1)^{\alpha(P)}\mathbb{1},$$

where $\alpha(P)$ is the number of projections in P which equal to $(\mathbb{1} - \Phi(d))$. Let us define an operator

$$F_n(w) = \sum_{P \in \mathcal{P}_n : \alpha(P) \leq w} P.$$

It is the sum of all possible n-fold tensor products with the number of $(\mathbb{1} - \Phi(d))$ elements in each P smaller than the parameter w. By calculating $F_n(w)^2$ we can verify that it is $F_n(w)$, thus $F_n(w)$ is a projection. It follows from the fact that operators P are drawn from $\mathcal{P}_n$ which was the set of mutually orthogonal projections. In this case

$$0 \leq F_n(w) \leq \mathbb{1}$$

which means that the operator $F_n(w)$ satisfies the first requirement from the Theorem (3.1). Since the fidelity of distillation is defined as

$$F_{\Gamma}(\rho; K) = \max_{F} \mathrm{Tr}(F\rho),$$

and we are interested in the isotropic state

$$I_d(f) := f\Phi(d) + \frac{1-f}{d^2-1}(\mathbb{1} - \Phi(d)),$$

and the distillable entanglement

$$D_{\Gamma}(\rho) := \sup\left\{ r : \lim_{n\to\infty} F_{\Gamma}(\rho^{\otimes n}; \lfloor 2^{rn} \rfloor) = 1 \right\},$$

we are interested in calculating $\mathrm{Tr}(F_n(w)I_d(f)^{\otimes n})$. First, we will evaluate the expression to be traced

$$F_n(w)I_d(f)^{\otimes n} = (\sum_{P \in \mathcal{P}_n : \alpha(P) \leq w} P)(f\Phi(d) + \frac{1-f}{d^2-1}(\mathbb{1} - \Phi(d)))^{\otimes n}.$$

The right factor will generate the weighted sum of all possible n-fold tensor products of $\Phi(d)$ and $(\mathbb{1} - \Phi(d))$

$$(f\Phi(d) + \frac{1-f}{d^2-1}(\mathbb{1} - \Phi(d)))^{\otimes n} = \sum_{P \in \mathcal{P}_n} f^{n-\alpha(P)}(\frac{1-f}{d^2-1})^{\alpha(P)} P.$$

Thus, we evaluate

$$(\sum_{P \in \mathcal{P}_n : \alpha(P) \leq w} P)(\sum_{P \in \mathcal{P}_n} f^{n-\alpha(P)}(\frac{1-f}{d^2-1})^{\alpha(P)} P) = \sum_{P \in \mathcal{P}_n : \alpha(P) \leq w} f^{n-\alpha(P)}(\frac{1-f}{d^2-1})^{\alpha(P)} P,$$

where we used the fact that projections P are idempotent and mutually orthogonal. To calculate the trace of the expression above, we note the following properties

$$\mathrm{Tr}(X \otimes Y) = \mathrm{Tr}(X)\,\mathrm{Tr}(Y),$$

$$\mathrm{Tr}(\Phi(d)) = 1,$$
$$\mathrm{Tr}(\mathbb{1} - \Phi(d)) = d^2 - 1.$$

Thus, for P with $\alpha(P)$ we have

$$\mathrm{Tr}(P) = \mathrm{Tr}(\mathbb{1} - \Phi(d))^{\alpha(P)}\,\mathrm{Tr}(\Phi(d))^{n-\alpha(P)} = (d^2-1)^{\alpha(P)}.$$

As a result, all P's which have the same $\alpha(P)$ will yield the same trace. To derive the trace of our sum, we have to do the counting of P's which have the same $\alpha(P)$. We can imagine that we have the set of $(\mathbb{1} - \Phi(d))$ projections $\{(\mathbb{1} - \Phi(d))_i\}$ where $1 \leq i \leq n$ indicate the position in the n-fold tensor product where the corresponding projection can be placed. Then, from that set of n projections we would like to obtain the subset of size $\alpha(P)$ and fill all of the remaining positions in the n-fold tensor product with projectors $\Phi(d)$. It can be done in $\binom{n}{\alpha(P)}$ ways. Therefore,

$$\mathrm{Tr}(\sum_{P\in\mathcal{P}_n:\alpha(P)\leq w} f^{n-\alpha(P)}(\frac{1-f}{d^2-1})^{\alpha(P)}P) = \sum_{0\leq i\leq w}\binom{n}{i} f^{n-i}(\frac{1-f}{d^2-1})^i(d^2-1)^i =$$

$$= \sum_{0\leq i\leq w}\binom{n}{i} f^{n-i}(1-f)^i.$$

We calculated that

$$\mathrm{Tr}(F_n(w)I_d(f)^{\otimes n}) = \sum_{0\leq i\leq w}\binom{n}{i} f^{n-i}(1-f)^i.$$

Recalling the definition of distillable entanglement D_Γ, we are interested in evaluating the trace above in the limit as $(n \to \infty)$ and understand under which conditions such a limit achieves the value of 1

$$\lim_{n\to\infty}\sum_{0\leq i\leq w}\binom{n}{i} f^{n-i}(1-f)^i = 1.$$

It can be noticed that the sum above is the partial sum of the discrete binomial distribution, thus the full sum yield the value of 1

$$\sum_{0\leq i\leq n}\binom{n}{i} f^{n-i}(1-f)^i = 1.$$

The partial sum can be expressed as a sum up to some fraction of n. Since we consider the limit as $(n \to \infty)$, we can observe that the sum has a value of 1 in the limit only when we sum up beyond the mean value of the binomial distribution which is $n(1-f)$ (the remaining tail will tend to 0). Thus, formally we can define the parameter ω and put the constraint on it in the following way

$$\omega := \lim_{n\to\infty}\frac{w}{n} > 1 - f.$$

Using bounds calculated earlier, we can put the following bound on $F_n(w)^\Gamma$,

$$|F_n(w)^\Gamma| \leq \sum_{P\in\mathcal{P}_n:\alpha(P)\leq w}|P^\Gamma| \leq \mathbb{1}d^{-n}\sum_{0\leq i\leq w}\binom{n}{i}(d+1)^i.$$

We recall from Theorem (3.1) that our operator must satisfy the following condition

$$|F_\infty(w)^\Gamma| \leq \frac{\mathbb{1}}{K} = \mathbb{1}2^{-rn},$$

where we skipped the floor function which is not relevant in the considered limit. Thus, assuming that $\omega < \frac{d+1}{d+2}$, we obtain the following lower bound on the distillable entanglement of the isotropic state $I_d(f)$:

$$r_{max} = D_\Gamma(I_d(f)) \geq -\lim_{n\to\infty}\frac{1}{n}\log_2\left[d^{-n}\sum_{0\leq i\leq w}\binom{n}{i}(d+1)^i\right] =$$

$$= -(-\omega\log_2\omega - (1-\omega)\log_2(1-\omega) + \omega\log_2(d+1) - \log_2 d) =$$

$$= \omega\log_2\frac{\omega}{d+1} + (1-\omega)\log_2(1-\omega) + \log_2 d,$$

with the constraint that $1-f < \omega < \frac{d+1}{d+2}$. To verify the limit above we can use the sandwich theorem with the following approximation of the binomial coefficient

$$\frac{2^{nH(\frac{m}{n})}}{n+1} \leq \binom{n}{m} \leq 2^{nH(\frac{m}{n})},$$

where $H(\frac{m}{n}) = -\frac{m}{n}\log_2(\frac{m}{n}) - (1-\frac{m}{n})\log_2(1-\frac{m}{n})$. Noting that our expression is monotonic in the specified range which is $1-f < \omega < \frac{d+1}{d+2}$, we can lower bound our limit by taking only the last term of the sum under the logarithm so that we have

$$\lim_{n\to\infty}\frac{1}{n}\log_2\left[\sum_{0\leq i\leq w}\binom{n}{i}(d+1)^i\right] \geq \lim_{n\to\infty}\left[\frac{1}{n}\log_2(\binom{n}{w}(d+1)^w)\right] \geq \lim_{n\to\infty}\left[\frac{1}{n}\log_2(\frac{2^{nH(\frac{m}{n})}}{n+1}(d+1)^w)\right] =$$

$$= \lim_{n\to\infty}\left[\frac{w\log_2(d+1)}{n} + H(\frac{w}{n}) - \frac{\log_2(n+1)}{n}\right] = \lim_{n\to\infty}\left[\omega\log_2(d+1) + H(\omega)\right] =$$

$$= \omega\log_2(d+1) - \omega\log_2(\omega) - (1-\omega)\log_2(1-\omega).$$

We can upper bound our limit by taking the last term multiplied by the number of terms under the logarithm so that we have

$$\lim_{n\to\infty}\frac{1}{n}\log_2\left[\sum_{0\leq i\leq w}\binom{n}{i}(d+1)^i\right] \leq \lim_{n\to\infty}\left[\frac{1}{n}\log_2(\binom{n}{w}(w+1)(d+1)^w)\right] \leq$$

$$\leq \lim_{n\to\infty}\left[\frac{1}{n}\log_2(2^{nH(\frac{m}{n})}(w+1)(d+1)^w)\right] = \lim_{n\to\infty}\left[\frac{\log_2(w+1)}{n} + H(\omega) + \frac{w}{n}\log_2(d+1)\right] =$$

$$= H(\omega) + \omega\log_2(d+1) = \omega\log_2(d+1) - \omega\log_2(\omega) - (1-\omega)\log_2(1-\omega).$$

Thus, we showed that our limit is bounded by 2 expressions which have the same limit which means that

$$\lim_{n\to\infty}\frac{1}{n}\log_2\left[d^{-n}\sum_{0\leq i\leq w}\binom{n}{i}(d+1)^i\right] = \omega\log_2(\frac{d+1}{\omega}) - (1-\omega)\log_2(1-\omega) - \log_2(d).$$

We observe that the bound is a decreasing function of ω in the range $1-f < \omega < \frac{d+1}{d+2}$. Thus, to obtain the best bound, we should consider the case when $\omega \to (1-f)$, then

$$D_\Gamma(I_d(f)) \geq (1-f)\log_2\frac{1-f}{d+1} + f\log_2(f) + \log_2 d.$$

4. Proof of Theorem (3.1)

Proof strategy

We will consider the definition of the fidelity of the K-state PPT entanglement distillation and simplify it by using the invariance when averaging over the Haar measure on unitaries. It will allow us to express the fidelity not as an optimization over PPT operation but as an optimization over Hermitian operators which meet certain conditions.

Proof

We recall the definition of the fidelity of the K-state PPT distillation for the state ρ

$$F_\Gamma(\rho;K) = \max_{\Psi} F(\Psi(\rho)).$$

Let us consider the operation Φ which maximizes the optimization above. We note that $F(\Psi(\rho))$ is invariant under composition of Φ with a unitary operator of the form $U \otimes \overline{U}$. If it holds for a single choice of the unitary operator, the invariance holds for the averaging over the Haar measure on $U(K)$ as well. Thus, we claim that our optimal $\Psi = T \circ \Psi$ i.e. Ψ is invariant under composition with the twirling operator T defined as such an average

$$T(X) = \int (U \otimes \overline{U}) X (U \otimes \overline{U})^* d\eta(U(K)).$$

The average above is a channel and is known in the literature as the isotropic twirling channel and has an alternative expression

$$T(X) = \mathrm{Tr}(X\Phi(K))\Phi(K) + \frac{1}{K^2-1}\mathrm{Tr}(X(\mathbb{1} - \Phi(K))(\mathbb{1} - \Phi(K)).$$

Its Choi representation is

$$J(T) = \Phi(K) \otimes \Phi(K) + \frac{1}{K^2-1}(\mathbb{1} - \Phi(K)) \otimes (\mathbb{1} - \Phi(K)).$$

We can also define a Choi-like representation as

$$J'(\Psi) = (\mathbb{1} \otimes \Psi)(\mathrm{vec}\,(\mathbb{1})\,\mathrm{vec}\,(\mathbb{1})^*),$$

as opposed to the typical Choi representation

$$J(\Psi) = (\Psi \otimes \mathbb{1})(\mathrm{vec}\,(\mathbb{1})\,\mathrm{vec}\,(\mathbb{1})^*).$$

Then by symmetry we see that

$$J(T) = J'(T).$$

We can calculate

$$J'(\Psi) = J'(T \circ \Psi) = (\Psi^* \otimes \mathbb{1})(J'(T)) = \Psi^*(\Phi(K)) \otimes \Phi(K) + \frac{1}{K^2-1}\Psi^*(\mathbb{1} - \Phi(K)) \otimes (\mathbb{1} - \Phi(K)) =$$

$$= H \otimes \Phi(K) + \frac{1}{K^2-1}(\mathbb{1} - H) \otimes (\mathbb{1} - \Phi(K)),$$

where we substituted $H = \Psi^*(\Phi(K))$ in the last step.
We can observe that $J'(\Psi)$ is positive if and only if $H \geq 0$ and $(\mathbb{1} - H) \geq 0$ (because $\Phi(K)$ is positive). To find the conditions on the partial transpose of H we consider

$$J'(\Psi)^\Gamma = H^\Gamma \otimes \Phi(K)^\Gamma + \frac{1}{K^2-1}(\mathbb{1} - H^\Gamma) \otimes (\mathbb{1} - \Phi(K)^\Gamma) =$$

$$= \frac{1}{K+1}\frac{(\frac{\mathbb{1}}{K} + H^\Gamma) \otimes (\mathbb{1} + K\Phi(K)^\Gamma)}{2} + \frac{1}{K-1}\frac{(\frac{\mathbb{1}}{K} - H^\Gamma) \otimes (\mathbb{1} - K\Phi(K)^\Gamma)}{2} =$$

$$= \frac{1}{K+1}\frac{(\frac{\mathbb{1}}{K} + H^\Gamma) \otimes (\mathbb{1} + W)}{2} + \frac{1}{K-1}\frac{(\frac{\mathbb{1}}{K} - H^\Gamma) \otimes (\mathbb{1} - W)}{2}$$

where we transformed the expression algebraically and then evaluated one of the partial transpositions to obtain the SWAP operator W. Projections $(\mathbb{1} + W)$ and $(\mathbb{1} - W)$ are positive operators, thus, the operator $J'(\Psi)^\Gamma$ is positive if and only if

$$-\frac{\mathbb{1}}{K} \leq H^\Gamma \leq \frac{\mathbb{1}}{K}.$$

Now, we recall the definition of fidelity F of the state ρ

$$F(\rho) = \mathrm{Tr}(\Phi(K)\rho).$$

Then, the fidelity of $\Psi(\rho)$ is

$$F(\Psi(\rho)) = \mathrm{Tr}(\Phi(K)\Psi(\rho)) = \langle \Psi^*\Phi(K), \rho\rangle = \langle H, \rho\rangle = \mathrm{Tr}(H^*\rho).$$

Since the fidelity of the K-state PPT distillation was defined as

$$F_\Gamma(\rho; K) = \max_\Psi F(\Psi(\rho)),$$

we conclude that

$$F_\Gamma(\rho; K) = \max_H \mathrm{Tr}(H\rho),$$

where H are Hermitian operators which satisfy

$$-\frac{\mathbb{1}}{K} \leq H^\Gamma \leq \frac{\mathbb{1}}{K},$$

and

$$0 \leq H \leq 1.$$